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Assessment of Bejan's heat exchanger allocation model under the influence of generalized thermal resistance, relaxation effect, bypass heat leak and internal irreversibility

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Abstract

This article reports an analytical investigation of the optimal heat exchanger allocation and the corresponding efficiency for maximum power output of a Carnot-like heat engine. To mimic a real engine, the generalized power law for the resistance in heat transfer external to the engine, relaxation effect in heat transfer, bypass heat leak and finally internal irreversibility of the power producing compartment of the engine is taken into consideration. From the engineering perspective the temperature ratio of the heat source and sink as well as to that of hot end and cold side of the working fluid is considered not to be the controllable parameters. A parametric study is presented for the other possible controllable variables. Selection of a power law over a linear model has a significant effect on the optimal heat exchanger allocation for maximum power output and the corresponding efficiency. For a higher degree of relaxation effect the drop in the maximum power efficiency is prominent along with the shift of equipartitioned allocation of heat exchanger inventory. Bypass heat leak and internal irreversibility exhibits relatively less pronounced effects on the maximum power efficiency and on the optimal heat exchanger allocation. Thus the endoreversible formulation of thermodynamic model is physically realistic. Strikingly when the optimal allocation of the heat exchanger inventory obeys the principle of equipartition in macroscopic organization for the linear law of the external heat resistance, the thermal efficiency appears to assume the representative documented value. Hence the linear model due to Bejan is also capable of capturing the essential features of a real power plant.

Keywords: Bypass heat leak; Carnot-like engine; Constructal; Equipartition; Heat exchanger allocation; Internal irreversibility; Power law heat transfer; Relaxation phenomenon; Synthetic constraint

1. Introduction

The fundamental contribution of simple models is to provide an estimate of different important parameters of a functional device and to establish a way for the more applied work that will follow in due course of industrial research and development [1]. In this present article many simplistic assumptions commonly adopted for a power plant will be abandoned making it more actual for the realistic performance. In the following paragraphs we discuss the issues that corners around the design of an actual power plant.

To start with there are many practical engineering concerns to contemplate in connection with the four-process model optimized by Curzon and Ahlborn [2]. Sadi Carnot's original essay [3] as interpreted graphically and analytically by Émile Clapeyron [4] is a description of a gas contained in a cylinder and piston mechanism that undergoes a cycle of four processes, two quasistatic and isothermal processes interspersed with two quasistatic and adiabatic processes. Landsberg et al. [5] generalized this cycle which is characterized by two adiabatics and two heat transfer paths with constant heat capacities. Curzon and Ahlborn added finite thermal resistances between the cylinder and the respective

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Nomenclature

a	arbitrary complex number
A_{H}	hot end heat exchanger surface area
A_{L}	cold end heat exchanger surface area
b	nonzero complex number
c_i	parametric constants for $i = 1, 2, 3, \ldots, 22$
C_i	internal thermal conductance of the power plant
n	power law index, Eqs. (3) or (4)
n_1	real part of complex power law index
n_2	imaginary part of complex power law index
P	power output of the engine, Eqs. (13) or (14)
$\overline{P}$	dimensionless power, Eqs. (20) or (21)
P_{r}	real part of dimensionless power, Eqs. (35) or (38)
$\dot{Q}'$	unsteady heat transfer rate
$\dot{Q}'_{\mathrm{HC}}$	unsteady heat transfer rate between work producing compartment and high temperature side, Eq. (3)
$\dot{Q}'_{\mathrm{LC}}$	unsteady heat transfer rate between work producing compartment and high temperature side, Eq. (3)
$\dot{Q}_{\mathrm{HC}}$	steady heat transfer rate between work producing compartment and high temperature side, Eq. (5)
$\dot{Q}_{\mathrm{L}}$	steady state heat rejected at the heat sink, Eq. (9)
$\dot{Q}_{\mathrm{LCE}}$	steady state heat released to the heat sink by endoreversible heat engine
$\dot{Q}_{\mathrm{LC}}$	steady heat transfer rate between work producing compartment and high temperature side, Eq. (6)
$\dot{Q}_i$	steady bypass heat leak through the machine structures, Eq. (7)
$\overline{\dot{Q}}_i$	nondimensional bypass heat leak
$\dot{Q}_{\mathrm{H}}$	steady state heat transfer rate supplied by the heat source, Eq. (8)
$\dot{S}_{\mathrm{gen}}$	entropy generation rate
T	temperature
ΔT	temperature as potential difference
T_{h}	real temperature component of the high temperature side, Eq. (1)
T_{H}	heat source temperature
T_{HC}	transient temperature of the working fluid at the hot end, Eq. (1)
T_{HO}	time averaged temperature of the working fluid at the hot end
T_{l}	real temperature component of the low temperature side, Eq. (2)
T_{L}	heat sink temperature
T_{LC}	transient temperature of the working fluid at the cold side, Eq. (2)
T_{LO}	time averaged temperature of the working fluid at the cold side
U_{H}	overall heat transfer rate of high temperature side heat exchanger
U_{L}	overall heat transfer rate of low temperature side heat exchanger
UA	overall thermal conductance, Eq. (17)
$\dot{W}$	work output rate
x	heat exchanger allocation ratio
x_{opt}	optimal heat exchanger allocation ratio
y	transformed heat exchanger allocation ratio, Eq. (46)
z	transformed heat exchanger allocation ratio, Eq. (50)
$\bar{z}$	transformed heat exchanger allocation ratio, Eq. (58)

Greek symbols

ϕ	degree of irreversibility, Eq. (10)
η	thermal efficiency of the engine, Eq. (15)
η_{r}	real part of the engine efficiency, Eq. (72)
τ	temperature ratio spanned by heat source and sink, Eq. (18)
τ_{h}	intermediate temperature ratio, Eq. (18)
τ_{o}	steady state temperature ratio spanned by working fluid, Eq. (18)
τ_{opt}	optimized temperature range of the working fluid
ω	oscillating periodic frequency of thermal wave, Eqs. (1) or (2)

Subscripts

H	quantities related to high temperature source
HC	transient quantities at the hot end
HO	time averaged quantities at the hot end
L	quantities related to low temperature sink
LO	time averaged quantities at the cold side
opt	optimized parameters
r	real part of a complex quantity
t	time average of quantities

Superscripts

–	dimensionless or transformed quantities

Symbols

$\langle\ \rangle$	represents a averaged quantity
Δ	represents a difference

temperature reservoirs and in this way described and optimized a more realistic time-dependent evolution of the cycle. The four-process model of Curzon and Ahlborn and its steady state counterpart introduced by Novikov

[6] and independently by Andresen et al. [7], Lu [8] and Bejan [9] was pursued along several lines. These were reviewed by Andresen et al. [10], Wu et al. [11], and Feidt et al. [12].

Adaptation of different nomenclatures related to this subject field is not to be confused with the maturity of this subject field. Terminology innovations included the introduction of the term "endoreversible" by Rubin [13] to describe the reversibility of the work producing compartment or alternatively the term "exoirreversible" for the external irreversibilities that surrounded the same compartment was mentioned by Radcenco [14]. It is to be noted that the concept of internal reversibility or external irreversibility is well established thermodynamic concept. It is tantamount to the local thermodynamic equilibrium model [15] that serves as the foundation for modern heat transfer and fluid dynamics [16]. The term "finite-time" thermodynamics was introduced by Andresen et al. [17] to describe the optimization of thermodynamic processes subjected to time constraints [18].

Again in involvement with actual engines we concern the following factors. The working spaces of many energy-conversion machines operate under the conditions of oscillating flow. These machines include Stirling engines and refrigerators, reciprocating internal combustion engines, gas and refrigerant compressors, cryocooler and expanders and compressors and pulse-tube refrigerators. Newton's law of cooling as a basis of typical convective heat transfer correlations states that heat transfer is proportional to the bulk gas and wall temperature difference. Fourier's law of conduction, an exact expression within the continuum hypothesis states that heat transfer is proportional to the temperature gradient at the wall. In most steady state convective heat transfer situations the wall temperature gradient is proportional to the bulk gas–wall temperature difference and so Newton's law works.

It is often neglected that Newton's law is an engineering approximation and heat transfer is not necessarily proportional to the bulk gas–wall temperature difference. In particular, Newton's law in its ordinary form is not valid in most oscillating pressure or oscillating flow heat transfer. There is a phase shift between heat transfer and temperature difference. So there are sections of the cycle where the conventional convective heat transfer coefficient becomes negative and points where it becomes infinite.

In several studies beginning with Gutkowicz-Krusin et al. [19] the assumption that the heat transfer rates are proportional to the local temperature difference were replaced by more general nonlinear heat transfer models that account for natural convection, radiation and temperature dependent properties by De Vos [20], Chen and Yan [21], Angulo-Brown and Paez-Hernandez [22]. Early studies were also conducted by Rubin [13], Lucca [23], Rozonoer and Tsirlin [24], Mozurkewich and Berry [25], and Tsirlin [26].

In circuit design with alternating electricity, circuit elements are assigned a complex impedance rather that real resistance. In mechanical vibration analysis, machine elements are assigned complex impedances rather than real inertias, damping constants and spring constants. In periodic conduction heat transfer problems and other similar differential equations are often solved using the method of complex field. This in essence reduces a periodic time-dependent problem in real variable to a steady state problem in complex variables. Once the complex solution has been found the results are usually given in terms of heat transfer magnitude and phase. The same information however, can be given in terms of real and imaginary parts of heat transfer. Expressing the thermal resistance of the boundary layer in terms of complex number is relatively an unemployed idea [27–29].

Next, the maximization of work output as opposed to power output was pursued by Grazzini and Gori [30] and Wu et al. [11]. The subtle differences between the maximum power in time dependent cyclic versus steady flow power plant models were clarified by Kiang and Wu [31]. As figure of merit in power plant optimization, Angulo-Brown [32] proposed to maximize the so-called ecological function $\dot{W} - T_{\mathrm{L}}\dot{S}_{\mathrm{gen}}$ where $\dot{W}$ is the work output rate, $\dot{S}_{\mathrm{gen}}$ is the entropy generation rate of the power plant and T_{L} is the heat sink temperature. Since in cases ecological function may assume negative values, recently Ust et al. [33] proposed to maximize the quantity $\frac{\dot{W}}{T_{\mathrm{L}}\dot{S}_{\mathrm{gen}}}$ known as ecological coefficient of performance.

Also several of these studies emphasized the importance of matching the temperature of the working fluid to the temperature of the heating agent. We can maximize the instantaneous power output of the model in two ways, with respect to the temperature range spanned by the working fluid (τ) and, or the allocation of the total thermal conductance (x). The practical implications of optimal allocation of heat exchanger inventory x_{opt} are clear and immediate-the heat exchanger inventory must be divided in a certain way. The practical implications of optimized temperature range of the working fluid τ_{opt} are more abstract. The message to the designer is that the working fluid must be selected in such a way it can be heated while at a certain temperature and be cooled at another optimal temperature for each given pair of heat exchanger inventories. The designer is considerably less free to experiment with the fluid type than to divide heat exchanger inventory. Another fact of the matter is that large scale power plants are optimized for fixed heat input and not variable heat input [34].

The objective of the current contribution is to provide an analytical model of a generalized power plant operating cyclically. In this study both the bypass heat leak and the internal irreversibility has been considered. Unsteady state heat transfer processes are modeled as complex. The conventional Newton's law of cooling has been replaced by a generalized power law. The exponent of the power law is also considered to be complex to include the relaxation process in heat transfer of the system [35–37]. Finally the optimization of power output [38,39] is carried out with respect to the optimal allocation of heat exchanger inventories [40–42] alone. The principle of operation of power plant at maximum power output over other objectives turned out to be the most natural choices as Odum and Pinkerton [43] furnished several examples of this category from the fields of engineering, physics and biology. In sum, the present exercise explores the range of feasibility of heat exchanger allocation model reported in Ref. [40] against more widely realistic conditions.

2. The physical model

With reference to Fig. 1 we discuss the following modeling features in succession. There remain many engineering problems for example heat transfer in combustion engine wall and space reentry problems in which boundary condition functions are time dependent. In nuclear reactor fuel elements during power transients, the energy generation rate varies with time. Duhamel's theorem [44] provides a convenient approach for developing solution to the heat conduction problems with time dependent boundary conditions and, or time dependent energy generation. Thus the external irreversibilities in heat transfer occurring at the hot end and the cold end heat exchangers between the heat engine and the corresponding thermal reservoir as considered by Curzon and Ahlborn [2] can be further modified from a more practical view of modeling.

On account of periodic heat transfer mechanism the temperatures of the heat source (T_{H}) and the heat sink (T_{L}) are different than the time averaged temperatures of the working fluid at the hot end (T_{HO}) and the cold side (T_{LO}), respectively. Thus the actual temperatures of the working fluid at the hot end and the cold side can be considered as the superposition of the steady part with the periodic part. So in transient form the temperatures of the working fluid at the hot end (T_{HC}) and the cold end (T_{LC}) can be expressed in complex combinations as

$$T_{\mathrm{HC}} = T_{\mathrm{HO}} + T_{\mathrm{H}} \exp(\mathrm{i}\omega t) \tag{1}$$

and

$$T_{\mathrm{LC}} = T_{\mathrm{LO}} + T_{\mathrm{L}} \exp(\mathrm{i}\omega t) \tag{2}$$

where T_{h} and T_{l} are some real quantities indicating temperatures, $\mathrm{i} = \sqrt{-1}$ and ω is the oscillating periodic frequency. The time-averaged quantities presented in Eqs. (1) and (2) assumes, respectively $\langle T_{\mathrm{HC}} \rangle_t = T_{\mathrm{HO}}$ and $\langle T_{\mathrm{LC}} \rangle_t = T_{\mathrm{LO}}$. It is to be noted that the actual thermal reservoir [45] temperatures are considered to be real and constant in absence of any periodicity. For the energy flow to occur the second law of thermodynamics demands that $T_{\mathrm{L}} < T_{\mathrm{LO}} < T_{\mathrm{HO}} < T_{\mathrm{H}}$.

Next, we relax another stipulation of Curzon–Ahlborn model [2] that is the heat transfer external to the endoreversible compartment is linear. Instead we adopt the fact that the unsteady state heat transfer rate ($\dot{Q}'$) between the work producing compartment of the engine and its surroundings follow a generalized power law for the temperature (T) of the form $\dot{Q}' \propto \Delta T^n$ where n is the index of the power law. Thus for the high temperature side of the heat engine we have

$$\dot{Q}'_{\mathrm{HC}} = U_{\mathrm{H}} A_{\mathrm{H}} (T_{\mathrm{H}}^n - T_{\mathrm{HC}}^n) \tag{3}$$

where U_{H} is the overall heat transfer coefficient based upon heat transfer surface area A_{H} of the hot end heat exchanger. Similarly, for the low temperature side of the heat engine we obtain

$$\dot{Q}'_{\mathrm{LC}} = U_{\mathrm{L}} A_{\mathrm{L}} (T_{\mathrm{LC}}^n - T_{\mathrm{L}}^n) \tag{4}$$

where U_{L} is the overall heat transfer coefficient and A_{L} is the heat transfer surface area of the cold end of the heat exchanger.

Now, we would like to incorporate the physical process of relaxation phenomena [35–37]. One way is to incorporate this feature is to consider the index of heat transfer power law to be imaginary that is $n = n_1 + \mathrm{i}n_2$ where n_1 and n_2 are both real quantities. The imaginary part corresponding to n_2 takes into account the relaxation phenomena in heat transfer mechanism. In abstract thinking it is to be understood that the imaginary part of the complex index brings about the complex nature of temperature and hence capable of incorporating the phenomenon of phase lag or equivalently relaxation phenomenon. It is further to be noted that for the negative values of the real part of the power law index of heat transfer, overall heat transfer coefficient will have negative values. Such an occurrence is tantamount to the negative heat capacity [46].

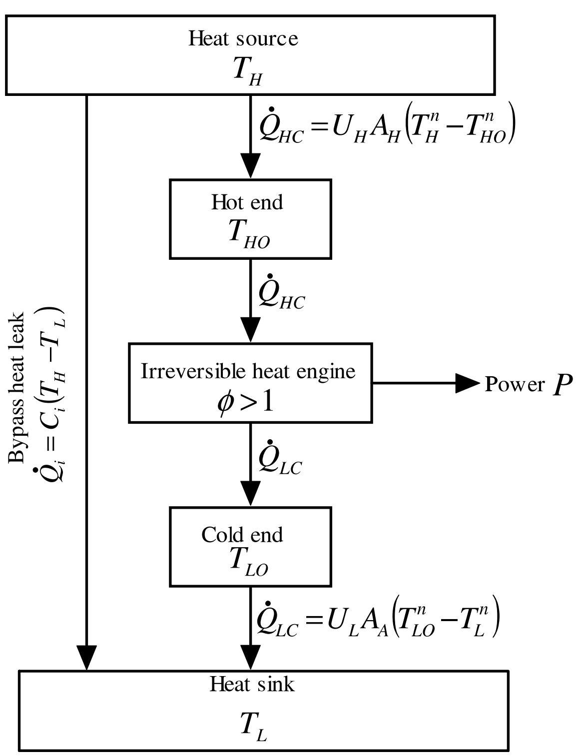

Fig. 1. Generalized irreversible heat engine with thermal resistance, bypass heat leak and internal irreversibility.

Again in view of steady state cyclic operation Eqs. (3) and (4) assumes a new form

$$\dot{Q}_{\mathrm{HC}} \approx U_{\mathrm{H}}A_{\mathrm{H}}(T_{\mathrm{H}}^{n} - T_{\mathrm{HO}}^{n}) \tag{5}$$

and

$$\dot{Q}_{\mathrm{LC}} = U_{\mathrm{L}}A_{\mathrm{L}}(T_{\mathrm{LO}}^{n} - T_{\mathrm{L}}^{n}) \tag{6}$$

where the time averaged quantities of heat transfer rates are $\langle \dot{Q}'_{\mathrm{HC}} \rangle_t = \dot{Q}_{\mathrm{HC}}$ and $\langle \dot{Q}'_{\mathrm{LC}} \rangle_t = \dot{Q}_{\mathrm{LC}}$, respectively. The physically realistic basis of approximation incurred in Eqs. (5) and (6) are respectively $\left|\frac{T_{\mathrm{H}}}{T_{\mathrm{HO}}}\exp(\mathrm{i}\omega t)\right| \ll 1$ or simply $\left|\frac{T_{\mathrm{H}}}{T_{\mathrm{HO}}}\right| \ll 1$ and $\left|\frac{T_{\mathrm{L}}}{T_{\mathrm{LO}}}\exp(\mathrm{i}\omega t)\right| \ll 1$ or simply $\frac{T_{\mathrm{L}}}{T_{\mathrm{LO}}} \ll 1$.

Another type of heat loss is due to wall heat loss. A host of power plant elements fall into this category. For example the heat lost through the wall of a combustion chamber or boiler house, heat removed by the cooling system of an internal combustion engine and the streamwise convective heat leak channeled towards room temperature by the counter flow heat exchanger of a regenerative Brayton cycle [47]. This modeling feature of heat loss is known as bypass heat leak and was first pointed out by Bejan and Paynter [48]. For this heat transfer rate $\dot{Q}_i$ to be constant which leaks directly through the machine structures and around the power producing compartment we assume

$$\dot{Q}_i = C_i(T_{\mathrm{H}} - T_{\mathrm{L}}) \tag{7}$$

where C_i is the shorthand for the internal thermal conductance of the power plant. Upon energy balance we arrive at the following heat transport equalities

$$\dot{Q}_{\mathrm{H}} = \dot{Q}_i + \dot{Q}_{\mathrm{HC}} \tag{8}$$

and

$$\dot{Q}_{\mathrm{L}} = \dot{Q}_i + \dot{Q}_{\mathrm{LC}} \tag{9}$$

where $\dot{Q}_{\mathrm{H}}$ is the heat transfer rate provided by the hot end thermal reservoir and $\dot{Q}_{\mathrm{L}}$ is the heat transfer rate rejected to the cold end of the thermal reservoir.

In reality the work producing compartment is also irreversible due to various nonequilibrium events inside the engine that ultimately produces the power. So finally we relax the imposed consideration of endoreversibility. This can be simply achieved by the introduction of a dimensionless factor. From the second law of thermodynamics an irreversible heat engine will release more heat than its counterpart reversible heat engine. If $\dot{Q}_{\mathrm{LC}}$ is the rate of heat flow released by the cold working fluid to the cold side heat exchanger and $\dot{Q}_{\mathrm{LCE}}$ is that of for the endoreversible heat engine, the degree of irreversibility ϕ can be defined as

$$\phi = \frac{\dot{Q}_{\mathrm{LC}}}{\dot{Q}_{\mathrm{LCE}}} \geqslant 1. \tag{10}$$

3. The optimization method

First we impose the restriction of endoreversibility for the work producing compartment. Then the second law of thermodynamics applied for this compartment relates the heat transfer and temperature quantities in the following manner

$$\frac{\dot{Q}_{\mathrm{HC}}}{T_{\mathrm{HO}}} - \frac{\dot{Q}_{\mathrm{LCE}}}{T_{\mathrm{LO}}} = 0. \tag{11}$$

In the next step the foregoing limitation is waived with the aid of degree of irreversibility concept introduced in Eq. (10). Thus the Eq. (10) in combination with Eq. (11) labels the work producing compartment to be irreversible. The corresponding relation between the heat transfer and temperature interaction assumes the transformation

$$\frac{\dot{Q}_{\mathrm{LC}}}{\dot{Q}_{\mathrm{HC}}} = \frac{\dot{Q}_{\mathrm{LC}}}{\dot{Q}_{\mathrm{LCE}}}\frac{\dot{Q}_{\mathrm{LCE}}}{\dot{Q}_{\mathrm{HC}}} = \phi\frac{T_{\mathrm{LO}}}{T_{\mathrm{HO}}}. \tag{12}$$

Next, the power output P of the engine is dictated by the first law of thermodynamics as

$$P = \dot{Q}_{\mathrm{H}} - \dot{Q}_{\mathrm{L}}. \tag{13}$$

There is a reduction of power output in presence of bypass heat leak phenomenon. Introducing Eqs. (8) and (9) into Eq. (13) we obtain

$$\begin{aligned} P &= \dot{Q}_{\mathrm{H}} - \dot{Q}_{\mathrm{L}} = (\dot{Q}_i + \dot{Q}_{\mathrm{HC}}) - (\dot{Q}_i + \dot{Q}_{\mathrm{LC}}) \\ &= \dot{Q}_{\mathrm{HC}} - \dot{Q}_{\mathrm{LC}}. \end{aligned} \tag{14}$$

The thermodynamic efficiency of the heat engine is defined as

$$\eta = \frac{P}{\dot{Q}_{\mathrm{H}}} = \frac{P}{\dot{Q}_i + \dot{Q}_{\mathrm{HC}}}. \tag{15}$$

Now, in order to limit the degrees of freedom of the physical model for the objective of optimization we look back into Eqs. (5) and (6). We postulate that the proportionality constants contained in Eqs. (5) and (6) compete with each other. From the physics of the problem the proportionality constants are the conductances of the respective heat exchangers and as such the commodities of short supply. Thus it makes perfect sense to recognize that the total conductance inventory as a constraint is also obeying the relation [41]

$$U_{\mathrm{H}}A_{\mathrm{H}} + U_{\mathrm{L}}A_{\mathrm{L}} = \mathrm{UA}. \tag{16}$$

Accounting for such a constraint of the type Eq. (16) is well established in literature and was also deduced employing method of synthetic constraint [49]. Among other choices such as area constraint [41], cost constraint [50] and the entropy generation rate constraint [51], the Eq. (16) is indicated to be the most natural selection [52]. Henceforth we will consider the conductances to be the single entity of the proportionality constants rather than the products of overall heat transfer coefficients and corresponding areas in Eqs. (5) and (6). Mathematically we mean $(\mathrm{UA})_{\mathrm{H}} = U_{\mathrm{H}}A_{\mathrm{H}}$ and $(\mathrm{UA})_{\mathrm{L}} = U_{\mathrm{L}}A_{\mathrm{L}}$ where UA itself stands for a single commodity. In terms of conductance allocation ratio x we may write

$$(\mathrm{UA})_{\mathrm{H}} = \left(\frac{x}{1+x}\right)\mathrm{UA},$$
$$(\mathrm{UA})_{\mathrm{L}} = \left(\frac{1}{1+x}\right)\mathrm{UA} \quad \text{and} \quad \frac{(\mathrm{UA})_{\mathrm{L}}}{(\mathrm{UA})_{\mathrm{H}}} = \frac{1}{x}. \tag{17}$$

Further we define various intermediate temperature ratios as

$$\tau = \frac{T_{\mathrm{L}}}{T_{\mathrm{H}}}, \quad \tau_{\mathrm{o}} = \frac{T_{\mathrm{LO}}}{T_{\mathrm{HO}}} \quad \text{and} \quad \tau_{\mathrm{H}} = \frac{T_{\mathrm{HO}}}{T_{\mathrm{H}}}. \tag{18}$$

Now from Eq. (12) employing Eqs. (5) and (6) and invoking nondimensional parameters defined in Eqs. (17) and (18) we obtain

$$\tau_{\mathrm{H}}^{n} = \frac{\phi x \tau_{\mathrm{o}}^{1-n} + \left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n}}{1 + \phi x \tau_{\mathrm{o}}^{1-n}}. \tag{19}$$

Similarly, with the aid of Eqs. (12), (5) and (6), the Eq. (14) for the power output transforms into

$$\overline{P} = (1 - \phi\tau_{\mathrm{o}})\left(\frac{x}{1+x}\right)(1 - \tau_{\mathrm{H}}^{n}) \tag{20}$$

where the dimensionless power $\overline{P}$ is defined as $\overline{P} = \frac{P}{\mathrm{UA}T_{\mathrm{H}}^{n}}$. The elimination of τ_{h} term between Eqs. (19) and (20) leads to the result

$$\overline{P} = (1 - \phi\tau_{\mathrm{o}})\left(\frac{x}{1+x}\right)\left[\frac{1 - \left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n}}{1 + \phi x \tau_{\mathrm{o}}^{1-n}}\right]. \tag{21}$$

Eq. (21) can be rearranged to obtain

$$\left(\frac{1 - \phi\tau_{\mathrm{o}}}{\overline{P}}\right)\left(\frac{x}{1+x}\right) = \frac{1 + \phi x \tau_{\mathrm{o}}^{1-n}}{1 - \left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n}}. \tag{22}$$

Now, deploying Eqs. (5) and (21) into Eq. (15) we obtain a revised expression for the efficiency as

$$\eta = \frac{1 - \phi\tau_{\mathrm{o}}}{1 + \bar{\dot{Q}}_{i}\left[\frac{1 + \phi x \tau_{\mathrm{o}}^{1-n}}{1 - \left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n}}\right]} \tag{23}$$

where the dimensionless bypass heat leak is denoted by $\bar{\dot{Q}}_{i} = \frac{\dot{Q}}{\mathrm{UA}T_{\mathrm{H}}^{n}}$.

Invoking the intermediate relation Eq. (22) into Eq. (23) we arrive at the more simplified expression for the efficiency as

$$\eta = \frac{1 - \phi\tau_{\mathrm{o}}}{1 + \bar{\dot{Q}}_{i}\left(\frac{x}{1+x}\right)\left(\frac{1 - \phi\tau_{\mathrm{o}}}{\overline{P}}\right)}. \tag{24}$$

A simple arrangement of Eq. (24) leads to the expression

$$\bar{\dot{Q}}_{i}\left(\frac{x}{1+x}\right)\frac{1}{\overline{\overline{P}}} - \frac{1}{\eta} = \frac{1}{\phi\tau_{\mathrm{o}} - 1}. \tag{25}$$

Eq. (25) is but an interesting result in thermodynamic optimization. When the parameters contained in this equation do not enjoy any degrees of freedom, we can find either the optimal efficiency for a given power output or the maximum power output for a given efficiency. The result found from these two extremal conditions are the same, so either of the two conditions may be used [21]. In a variational formulation of the problem of this type [21] the situation represents a transversality condition [53]. From the physics of the relation, Eq. (25) represents a competition between the power output and the corresponding efficiency of the engine. Thus in another sense Eq. (25) is a variant of synthetic constraint introduced as an optimization philosophy [49].

Since the index n of the power law is indicated to be a complex quantity, the expression for the power output obtained from Eq. (21) represents a complex number. However, the magnitude of the power output is the real part of that complex number. So we proceed to extract the real part of the power in the following manner. If b be a nonzero complex number, and let a be a completely arbitrary complex number, then by the general power b^{a} we mean every value given by the formula $b^{a} = \mathrm{e}^{a \ln b}$ [54]. Thus we have for

$$\begin{aligned}\left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n} &= \left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n_1 + \mathrm{i}n_2} \\ &= \left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n_1}\cos\left[n_2 \ln\left(\frac{\tau}{\tau_{\mathrm{o}}}\right)\right] + \mathrm{i}\left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n_1}\sin\left[n_2 \ln\left(\frac{\tau}{\tau_{\mathrm{o}}}\right)\right]\end{aligned} \tag{26}$$

and

$$\begin{aligned}\tau_{\mathrm{o}}^{1-n} &= \tau_{\mathrm{o}}^{(1-n_1) - \mathrm{i}n_2} \\ &= \tau_{\mathrm{o}}^{1-n_1}\cos[n_2 \ln(\tau_{\mathrm{o}})] - \mathrm{i}\tau_{\mathrm{o}}^{1-n_1}\sin[n_2 \ln(\tau_{\mathrm{o}})].\end{aligned} \tag{27}$$

So the expression contributing to the complex number in Eq. (21) is expressed as

$$\begin{aligned}\frac{1 - \left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n}}{1 + \phi x \tau_{\mathrm{o}}^{1-n}} &= \frac{c_1 - \mathrm{i}c_2}{c_4 - \mathrm{i}c_6} \\ &= \left(\frac{c_1 c_4 + c_2 c_6}{c_4^2 + c_6^2}\right) + \mathrm{i}\left(\frac{c_1 c_6 - c_2 c_4}{c_4^2 + c_6^2}\right)\end{aligned} \tag{28}$$

where

$$c_1 = 1 - \left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n_1}\cos\left[n_2 \ln\left(\frac{\tau}{\tau_{\mathrm{o}}}\right)\right] \tag{29}$$

$$c_2 = \left(\frac{\tau}{\tau_{\mathrm{o}}}\right)^{n_1}\sin\left[n_2 \ln\left(\frac{\tau}{\tau_{\mathrm{o}}}\right)\right] \tag{30}$$

$$c_3 = \phi\tau_{\mathrm{o}}^{1-n_1}\cos[n_2 \ln(\tau_{\mathrm{o}})] \tag{31}$$

$$c_4 = 1 + c_3 x \tag{32}$$

$$c_5 = \phi\tau_{\mathrm{o}}^{1-n_1}\sin[n_2 \ln(\tau_{\mathrm{o}})] \tag{33}$$

and

$$c_6 = c_5 x \tag{34}$$

Thus the real part of the power delivered assumes the form

$$P_{\mathrm{r}} = \mathrm{Re}(\overline{P}) = (1-\phi\tau_{\mathrm{o}})\left(\frac{x}{1+x}\right)\left(\frac{c_7x+c_1}{c_8x^2+2c_3x+1}\right). \quad (35)$$

where P_{r} stands for the real part of the power and the parameters are

$$c_7 = c_1c_3 + c_2c_5 \quad (36)$$

and

$$c_8 = c_3^2 + c_5^2. \quad (37)$$

Eq. (35) may further be rearranged as

$$P_{\mathrm{r}} = (1-\phi\tau_{\mathrm{o}})\left(\frac{c_7x^2+c_1x}{c_8x^3+c_9x^2+c_{10}x+1}\right) \quad (38)$$

where

$$c_9 = 2c_3 + c_8 \quad (39)$$

and

$$c_{10} = 2c_3 + 1. \quad (40)$$

Now, we return back to our fundamental objective of optimal heat exchanger allocation for which power output is maximum. Thus setting the first derivative equal to zero of Eq. (38) with respect to the optimal heat exchanger allocation ratio x we obtain

$$x^4 + c_{11}x^3 + c_{12}x^2 + c_{13}x + c_{14} = 0 \quad (41)$$

where

$$c_{11} = 2\frac{c_1}{c_7} \quad (42)$$

$$c_{12} = \frac{c_1c_9 - c_7c_{10}}{c_7c_8} \quad (43)$$

$$c_{13} = -2\frac{1}{c_8} \quad (44)$$

and

$$c_{14} = -\frac{c_1}{c_7c_8}. \quad (45)$$

The quartic Eq. (41) is a commonplace occurrence in many physical problems and can be solved completely analytically in the following steps. The algebraic form in which the solutions of the quartic equation normally appear is so awkward and clumsy that they are of little use for further manipulations in obtaining closed form expressions of other related quantities. Thus we proceed to furnish an elegant expression for the optimal heat exchanger allocation ratio. The solution methodology is central to such category of optimization problem.

First we reduce the quartic equation into a cubic equation on following Descartes' method [55]. Substituting $x = y - \frac{1}{4}c_{11}$ in Eq. (41) we have the transformed equation

$$y^4 + c_{15}y^2 + c_{16}y + c_{17} = 0. \quad (46)$$

where

$$c_{15} = c_{12} - \frac{3}{8}c_{11}^2 \quad (47)$$

$$c_{16} = c_{13} + \frac{1}{8}c_{11}^3 - \frac{1}{2}c_{11}c_{12} \quad (48)$$

and

$$c_{17} = c_{14} - \frac{3}{256}c_{11}^4 + \frac{1}{16}c_{11}^2c_{12} - \frac{1}{4}c_{11}c_{13}. \quad (49)$$

Now, the solution of the transformed quartic Eq. (46) can be obtained from the following auxiliary cubic equation

$$z^3 + c_{18}z^2 + c_{19}z + c_{20} = 0 \quad (50)$$

where

$$c_{18} = \frac{1}{2}c_{15} \quad (51)$$

$$c_{19} = \frac{1}{16}(c_{15}^2 - 4c_{17}) \quad (52)$$

and

$$c_{20} = -\frac{1}{64}c_{16}^2. \quad (53)$$

If z_1, z_2 and z_3 are the three roots of the Eq. (50), then the four roots of the transformed quartic equation in y are $\pm\sqrt{z_1} \pm \sqrt{z_2} \pm \sqrt{z_3}$. So the solutions of the original equation in x are

$$x_1 = \sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3} - \frac{1}{4}c_{11} \quad (54)$$

$$x_2 = \sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3} - \frac{1}{4}c_{11} \quad (55)$$

$$x_3 = -\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3} - \frac{1}{4}c_{11} \quad (56)$$

and

$$x_4 = -\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3} - \frac{1}{4}c_{11}. \quad (57)$$

Next, we are concerned about the solution of the cubic Eq. (50). We adopt the procedure outlined by Mckelvey [56]. Substituting $z = \bar{z} - \frac{1}{3}c_{18}$ in Eq. (50) we obtain

$$\bar{z}^3 \pm 3c_{21}\bar{z} + 2c_{22} = 0 \quad (58)$$

where

$$\pm 3c_{21} = c_{19} - \frac{1}{3}c_{18}^2 \quad (59)$$

and

$$2c_{22} = \frac{2}{27}c_{18}^3 - \frac{1}{3}c_{18}c_{19} + c_{20}. \quad (60)$$

Then the roots of the Eq. (50) can be listed as follows [56]:

Case I: When $c_{21} > 0$, one real root and two complex roots are

$$z_1 = -2\sqrt{c_{21}}\sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right] - \frac{1}{3}c_{18} \tag{61}$$

$$z_2 = \sqrt{c_{21}}\left\{\sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right] + i\sqrt{3}\cosh\left[\frac{1}{3}\sinh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right]\right\} - \frac{1}{3}c_{18} \tag{62}$$

and

$$z_3 = \sqrt{c_{21}}\left\{\sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right] - i\sqrt{3}\cosh\left[\frac{1}{3}\sinh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right]\right\} - \frac{1}{3}c_{18}. \tag{63}$$

Case II: When $c_{21} < 0$ and $c_{22}^2 - c_{21}^3 > 0$, one real root and two complex roots are

$$z_1 = -2\sqrt{c_{21}}\cosh\left[\frac{1}{3}\cosh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right] - \frac{1}{3}c_{18} \tag{64}$$

$$z_2 = \sqrt{c_{21}}\left\{\cosh\left[\frac{1}{3}\cosh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right] + i\sqrt{3}\sinh\left[\frac{1}{3}\cosh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right]\right\} - \frac{1}{3}c_{18} \tag{65}$$

and

$$z_3 = \sqrt{c_{21}}\left\{\cosh\left[\frac{1}{3}\cosh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right] - i\sqrt{3}\sinh\left[\frac{1}{3}\cosh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right]\right\} - \frac{1}{3}c_{18}. \tag{66}$$

Case III: When $c_{21} < 0$ and $c_{22}^2 - c_{21}^3 < 0$, three real roots are

$$z_1 = -2\sqrt{c_{21}}\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right] - \frac{1}{3}c_{18} \tag{67}$$

$$z_2 = \sqrt{c_{21}}\left\{\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right] + \sqrt{3}\sin\left[\frac{1}{3}\cos^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right]\right\} - \frac{1}{3}c_{18} \tag{68}$$

and

$$z_3 = \sqrt{c_{21}}\left\{\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right] - \sqrt{3}\sin\left[\frac{1}{3}\cos^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)\right]\right\} - \frac{1}{3}c_{18}. \tag{69}$$

In order to calculate the roots of the Eq. (41) via Eq. (61) through Eq. (69) it is convenient to employ the following relations [57]

$$\sinh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right) = \ln\left[\frac{c_{22}}{c_{21}^{3/2}} + \sqrt{\left(\frac{c_{22}}{c_{21}^{3/2}}\right)^2 + 1}\right] \tag{70}$$

and

$$\cosh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right) = \pm\ln\left[\frac{c_{22}}{c_{21}^{3/2}} + \sqrt{\left(\frac{c_{22}}{c_{21}^{3/2}}\right)^2 - 1}\right]. \tag{71}$$

Eq. (70) is numerically robust for $\frac{c_{22}}{c_{21}^{3/2}} \geqslant 0$. For negative values of $\frac{c_{22}}{c_{21}^{3/2}}$, we utilize the symmetry property $\sinh^{-1}\left(-\frac{c_{22}}{c_{21}^{3/2}}\right) = -\sinh^{-1}\left(\frac{c_{22}}{c_{21}^{3/2}}\right)$. It is to be noted that Eq. (71) is valid only for $\frac{c_{22}}{c_{21}^{3/2}} \geqslant 1$.

Thus we obtain an exact analytical expression for the real part of the power from Eq. (35). Then the real part of the efficiency is calculated from Eq. (25) using Eq. (35) as

$$\eta_r = Re(\eta) = \frac{1}{\frac{1}{1-\phi\tau_o} + \dot{Q}_i\left(\frac{x}{1+x}\right)\frac{1}{P_r}} \tag{72}$$

where η_r is the real part of the efficiency.

4. Numerical examples

The analytical results obtained in the foregoing section can be utilized to quantify the effects of various parameters on the allocation of heat exchanger inventory and the thermal efficiency of a real heat engine. In performing the numerical analysis of the physical model presented here, we adopt closely the following data from the existing power plants available in open literature [58–60].

From the physical point of view, the temperatures of the heat source and sink are considered not controllable. Similarly, the temperatures of the hot end and the cold side of the working fluid are also considered to be fixed for practical reasons. Thus in this parametric study we assume $\tau = \frac{2}{5}$ and $\tau_o = \frac{2}{3}$ not to vary. The discrete variations of other parameters considered are as follows: $n_1 = 1.0$, -1.0 and 4.0; $n_2 = 0.0$, 0.005, 0.025 and 0.125; $\phi = 1.0$, 1.005, 1.010 and 1.015 and $\dot{Q}_i = 0.001, 0.005, 0.025$ and 0.125. It is to be remarked that $n_1 = 1.0$ represents conventional Newton's law of convective cooling, whereas $n_1 = -1.0$ demonstrates the phenomenological heat transfer and $n_1 = 4.0$ categorizes the radiative heat transfer mode.

Table 1 shows the alteration of heat exchanger allocation ratio and the efficiency with the variation of the real part of the power law index n_1 when the other parameters assume some representative values. It is observed that n_1

Table 1
Effect of power law on the heat exchanger allocation and the thermal efficiency of the engine

S.No.	τ	τ_o	n_1	n_2	ϕ	$\dot{Q}_i$	x	η_r
1.	$\frac{2}{5}$	$\frac{2}{3}$	1.0	0.0	1.0	0.025	1.0	0.296296269
2.	$\frac{2}{5}$	$\frac{2}{3}$	−1.0	0.0	1.0	0.025	1.50000012	0.355555534
3.	$\frac{2}{5}$	$\frac{2}{3}$	4.0	0.0	1.0	0.025	0.544331133	0.308217049

has major impact both on the heat exchanger allocation ratio and on the thermal efficiency of the power plant. It is interesting to report that optimal heat exchanger allocation ratio conforms to the macroscopic organization with equipartition principle when the law of heat resistance is linear that is for $n_1 = 1.0$. On the other hand for the nonlinear laws of heat transfer, the principle of equipartition does not hold any longer. For the phenomenological heat transfer law that is for $n_1 = -1.0$, hot end side has almost 60% and the cold end has nearly 40% of the total heat exchanger inventory. Again for the radiative mode of heat transfer law that is for $n_1 = 4.0$ we allocate 35% to the hot side and 65% to the cold side out of total heat exchanger inventory available. Also it is to be noticed that there is a gain in the efficiency of the engine in departing from the linear law of external heat transfer resistance with a subsequent deviation from the equipartitioned allocation of heat exchanger inventory.

Table 2 represents the effect of relaxation in heat transfer both on the heat exchanger allocation and the efficiency of the engine. It is observed that when the parameter n_2 is high indicting a prominent effect of relaxation, the heat exchanger allocation ratio departs from the equipartition principle and there is a subsequent drop in efficiency of the engine. The higher the magnitude of relaxation effect in heat transfer the greater is the deviation from equal allocation of heat exchanger and the lesser is the thermal efficiency.

Table 3 reveals the effect of irreversibility of the work producing compartment. It is seen that within the range of variation of the parameter ϕ considered, the optimal heat exchanger allocation ratio drops slightly to nearly follow the equipartition in macroscopic organization and in correspondence there remains a little sparing of the thermal efficiency of the heat engine.

Table 4 demonstrates the influence of bypass heat leak on the heat exchanger allocation ratio and the thermal efficiency of the engine. It is found that bypass heat leak is not sensitive to the optimal heat exchanger allocation ratio but the thermal efficiency of the engine is affected. Optimal heat exchanger inventory distribution closely follows the macroscopic organization with the principle of equipartition. The stronger is the effect of bypass heat leak the greater is the drop in the thermal efficiency of the heat engine.

Table 2
Effect of relaxation on the heat exchanger allocation and the thermal efficiency of the engine

Sl. no.	τ	τ_o	n_1	n_2	ϕ	$\bar{\dot{Q}}_i$	x	η_r
1	$\frac{2}{5}$	$\frac{2}{3}$	1.0	0.005	1.0	0.025	0.997458279	0.296254754
2	$\frac{2}{5}$	$\frac{2}{3}$	1.0	0.025	1.0	0.025	0.987142146	0.296092391
3	$\frac{2}{5}$	$\frac{2}{3}$	1.0	0.125	1.0	0.025	0.932174623	0.295382202

Table 3
Effect of internal irreversibility on the heat exchanger allocation and the thermal efficiency of the engine

Sl. no.	τ	τ_o	n_1	n_2	ϕ	$\bar{\dot{Q}}_i$	x	η_r
1	$\frac{2}{5}$	$\frac{2}{3}$	1.0	0.025	1.005	0.025	0.984683096	0.293090314
2	$\frac{2}{5}$	$\frac{2}{3}$	1.0	0.025	1.010	0.025	0.982242286	0.29008919
3	$\frac{2}{5}$	$\frac{2}{3}$	1.0	0.025	1.015	0.025	0.979819536	0.28708902

Table 4
Effect of bypass heat leak on the heat exchanger allocation and the thermal efficiency of the engine

Sl. no.	τ	τ_o	n_1	n_2	ϕ	$\bar{\dot{Q}}_i$	x	η_r
1	$\frac{2}{5}$	$\frac{2}{3}$	1.0	0.025	1.0	0.001	0.987142146	0.331664711
2	$\frac{2}{5}$	$\frac{2}{3}$	1.0	0.025	1.0	0.005	0.987142146	0.325154096
3	$\frac{2}{5}$	$\frac{2}{3}$	1.0	0.025	1.0	0.125	0.987142146	0.204640388

5. Conclusions

In this study an analytical model of a Carnot-like heat engine in presence of power law external thermal resistance, relaxation effect in heat transfer, bypass heat leak and internal irreversibility has been presented. The thermal efficiency of the heat engine with the objective function of maximum power output investigated under the influence of various parameters.

It is observed that the power output and the thermal efficiency of the heat engine competes with each other and thus supports a body of observations in the theory of finite-time thermodynamics [21,61]. This competition is also the very philosophy of method of synthetic constraint introduced in optimization methodology [49].

For practical reasons the very engineering quest of optimal heat exchanger allocation for maximum power output is attended leaving behind the influence of optimal intermediate temperature ratio of the hot end and the cold side of the working fluid and also the effect of heat sink and heat source temperature ratio. It is noted that for optimal allocation of the heat exchanger inventory which maintains a category of macroscopic organization with the principle of equipartition, the efficiency at maximum power output also tends to assume a representative value [62].

The optimal heat exchanger allocation and the maximum power efficiency both are drastically affected by the selection of power law for the external heat transfer resistance. For a choice of radiative mode of power law for the external heat transfer, there is a great deviation from equipartitioned allocation of heat exchanger equipment but with a subsequent increase in the efficiency at maximum power. For an enhanced effect of relaxation phenomenon in heat transfer the optimal heat exchanger allocation ratio deviates relatively to a small extent and induces a negligible drop in the maximum power efficiency. The very effect of internal irreversibility is not that prominent on the heat exchanger allocation ratio and the maximum power efficiency. Thus the endoreversibility is a solid concept in the finite-time thermodynamic formulation of thermal systems. The bypass heat leak being diffused through the mechanical support only is considered to be linear and thus renders a noticeable effect on the maximum power efficiency for a relatively high value of the concerned parameter leaving the optimal heat exchanger allocation ratio to be near the equipartitioned value.

Though the allocation of heat exchanger inventory has its beginning with the work [9] it has been attacked from many different view points till recent time. The present contribution documents the most comprehensive treatment of a modeling philosophy of contemporary interest. It has been established that the second law analysis [63] if performed in proper fashion is worthy as always [64]. It is to be noted that the qualitative resemblance of heat exchanger distribution is independent of the type of energy source and application [65]. The philosophy of the engineered system remains to be unique and the same—the optimal distribution of imperfections. Such a tendency of optimized systems goes by the name of constructal law [66]. The future evolution of finite-time thermodynamics should follow this direction.

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